NON-SEPARATED CUTTINGS OF CONNECTED POINT SETS*

BY

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1. We shall consider a connected, metric and separable space which we denote by M. A subset X of M is called a cutting of M provided that the complement $M-X$ of X is not connected and hence is the sum of two mutually separated sets $M_1(X)$ and $M_2(X)$; X is said to separate two points or point sets A and B in M when the sets $M_1(X)$ and $M_2(X)$ can be so chosen that $M_1(X) \supset A$ and $M_2(X) \supset B$, and is said to separate a single set N in M when $M_1(X)$ and $M_2(X)$ can be chosen so that $N \cdot M_1(X) \neq 0 \neq N \cdot M_2(X)$.

A collection G of subsets of M will be called *non-separated* provided that the elements of G are mutually exclusive and no element of G separates any other element of G in M.

A subset P of M is said to have the potential order α in M relative to a given collection G of subsets of M provided that α is the least cardinal number such that there exists a monotonic decreasing sequence $[U_i]$ of neighborhoods of P such that $P=\prod_1^\infty \overline{U}_i$ and such that for each i, the boundary $F(U_i)$ of U_i is a subset of the sum of α of the sets of the collection G.

In this paper we shall show, first, that if G is any uncountable non-separated collection of cuttings of M then *all save a countable number of the elements of G have the potential order* 2 *in M relative to G.* Now obviously if the elements of any collection G of mutually exclusive cuttings of M are connected or if they reduce to single points, then the collection G is non-separated. And since for the case where M is compact, the potential order of a point of M is the same as its order in the Menger-Urysohn sense, our theorem yields as corollaries many important known results concerning the cut points and connected cuttings of connected sets and of continua; for example: (1) the theorem of Wazewski-Menger‡ that the ramification points of any acyclic continuous curve are countable, (2) the theorem of Kuratowski and Zarankiewicz§ that the set of all points of any connected set M whose complement in M is neither connected nor the sum of two connected point

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‡ See Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 49; and Menger, Fundamenta Mathematicae, vol. 10 (1927), p. 108.

§ Bulletin of the American Mathematical Society, vol. 33 (1927), p. 571.

sets is countable; (3) the theorem of the author* that all save a countable number of the cut points of any continuum are points of order 2 of M in the Menger-Urysohn sense; and (4) other results concerning cuttings due to Zarankiewicz† and to the author.‡

Second, with the aid of this theorem we shall show that if the space M contains an uncountable non-separated collection G of cuttings, then there exists an upper semi-continuous collection S of elements such that all save a countable number of the sets of G are elements of S and such that every two elements of S may be separated in M by some third element. In case M is compact, the decomposition space S is an acyclic continuous curve.

Finally, we shall prove an existence theorem to the effect that *every locally connected space M contains an uncountable non-separated collection of cuttings.* Therefore, the above mentioned decomposition is always realisable for locally connected sets M, and notably for the case where M is a continuous curve, this decomposition gives rise to a decomposition space which is a non-degenerate acyclic continuous curve.

2. **Preliminary lemmas.** Let X and Y be any two cuttings of M and set

$$\text{(i)} \qquad M - X = M_1(X) + M_2(X),$$

$$\text{(ii)} \qquad M - Y = M_1(Y) + M_2(Y),$$

representing decompositions of $M-X$ and $M-Y$ respectively into mutually separated sets. Then if i, j, r, and s are positive integers such that $i+j=3=r+s$, it follows immediately that the following equation is valid:

$$(2.1) \qquad M = M_i(X) + M_r(Y) + M_j(X)\cdot M_s(Y) + X + Y.$$

With the aid of this equation, we deduce at once the result

(2.2) *If neither of the sets X and Y separates the other, we may choose the indices i and r such that*

$$\text{(a)} \qquad X \subset M_r(Y) \text{ and } Y \subset M_i(X);$$

and these relations imply also the relations

$$\text{(b)} \qquad M_j(X)\cdot M_s(Y) = 0,\ M_j(X) + X \subset M_r(Y), \text{ and } M_s(Y) + Y \subset M_i(X).$$

Clearly this is the case, because by virtue of the relations (a) we may omit the last two terms in equation (2.1); and since M is connected, the term $M_j(X)\cdot M_s(Y)$ must vanish. This fact gives at once the remaining two relations (b).

* These Transactions, vol. 30 (1928), p. 606.

† See Fundamenta Mathematicae, vol. 12 (1928), pp. 119–125.

‡ See Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 87–104.

Now let G be any non-separated collection of cuttings of M and let $E(a, b)$ be the collection of all those elements of G which separate two given points a and b in M. Let X and Y be any two elements of $E(a, b)$ and let the indices in (i) and (ii) be chosen so that

(iii) $$M_1(X) \cdot M_1(Y) \supset a \text{ and } M_2(X) \cdot M_2(Y) \supset b.$$

The element X of $E(a, b)$ will be said to precede the element Y, and this fact is indicated by the notation $X < Y$, provided that for at least one set of decompositions satisfying (i), (ii) and (iii) it is true that $X \subset M_1(Y)$. We shall now show that this definition gives a natural order to the elements of $E(a, b)$.

First, for any two elements X and Y of $E(a, b)$, at least one of the relations $X < Y$ and $Y < X$ must be valid. For if X does not precede Y, then by (2.2), (a), $r = 2$ and hence $s = 1$. By (b) and (iii) it follows that $j = 2$ and hence $i = 1$. Therefore by (a), $Y \subset M_1(X)$, which means $Y < X$.

Second, only one of the relations $X < Y$ and $Y < X$ can be valid. For if $X < Y$, then [for any set of decompositions whatever satisfying (i), (ii), (iii)], in (2.2), $r = 1$ and hence $s = 2$. By (b) and (iii) it follows that $j = 1$ and hence $i = 2$. Therefore by (a), $Y \subset M_2(X)$, which is incompatible with $Y < X$.

Finally, for any three elements Z, X and Y of $E(a, b)$, the relations $Z < X$, $X < Y$ imply that $Z < Y$. For then $Z \subset M_1(X)$ and $X \subset M_1(Y)$. Hence in (2.2), $r = 1$ and $s = 2$. By (b) and (iii) it follows that $j = 1$. Therefore by the second relation in (b), $Z \subset M_1(X) + X \subset M_1(Y)$, which gives $Z < Y$.

Thus we have proved the following result:

(2.3) *If each element of the non-separated collection $E(a, b)$ of subsets of M separates the two points a and b in M, then the collection $E(a, b)$ possesses a natural order.*

For convenience we give here a lemma concerning ordered sets due to Zarankiewicz* which will be used below.

LEMMA (Zarankiewicz). *If K is any ordered subset of M, then the set H of all points p of K such that p is not at the same time a limit point of the set P_p of all points of K preceding p and also of the set F_p of all points of K following p is countable.*

The space M being separable and metric, it therefore contains a countable sequence $R_1, R_2, R_3, \cdots$ of open sets which is equivalent to the set of all open subsets of M. Now let H_1 be the set of all points of K which are not limit points of their predecessors, and let $H_2 = H - H_1$. For each point p of H_1 let $n(p)$ be the least positive integer such that $R_{n(p)}$ contains p but contains no point of K which precedes p. Then if p and q are distinct points of H_1 and

* See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

$p<q$, then since $R_{n(q)}$ does not contain p, it follows that $n(p)\neq n(q)$, and therefore H_1 is countable. A similar argument proves H_2 countable; and hence H is countable.

3. THEOREM. *If G is any uncountable non-separated collection of cuttings of a connected, metric, and separable space M, then all save possibly a countable number of the elements of G have the potential order 2 in M relative to G.*

Suppose, on the contrary, that G contains an uncountable subcollection G_1 no element of which has the potential order 2 in M relative to G. Now there exist two points a and b of M such that the collection $E(a, b)$ of all those elements of G_1 which separate a and b in M is uncountable; for M being separable, there exists a countable subset D of M such that $\bar{D}=M$; and since every element of G_1 which contains no point of D must separate some pair of points of D in M, and since the set of all pairs of points of D is countable, it follows that for at least one pair of points a, b of D, the set $E(a, b)$ is uncountable.

By §2 the elements of the collection $E(a, b)$ possess a natural order; and if K is a point set which contains exactly one point x of each element X of $E(a, b)$ and contains no other points, then K is an ordered point set. Indeed for each pair x, y of points of K, set $x<y$ provided that $X<Y$. By the Zarankiewicz lemma, the set H of all points p of K which are not at the same time a limit point both of their predecessors and of their successors is countable. Let $H(a, b)$ be the collection of all those sets X of $E(a, b)$ such that the corresponding point x in K belongs to $K-H$. Then $H(a, b)$ is uncountable and each element X of $H(a, b)$ contains a point x which is a limit point of the sum of the predecessors of X and also of the sum of the successors of X.

Now for each element X of $H(a, b)$, there exist mutually separated sets $M_1(X)$ and $M_2(X)$ such that

$$M - X = M_1(X) + M_2(X),\ M_1(X) \supset a \text{ and } M_2(X) \supset b.$$

And with the aid of what has just been shown it follows immediately that there exist two infinite sequences of elements $X_1, X_2, X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$ of $H(a, b)$ such that, for each n,

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n, \tag{1}$$

and such that X contains a point which is a limit point both of $\sum X_n$ and $\sum Y_n$.

Since by supposition no element of $H(a, b)$ can have the potential order 2 in M relative to G, it follows that if for each element X of $H(a,b)$, $V_n(X)$ denotes the set of points $M-[M_1(X_n)+M_2(Y_n)]$, then there exists at least one point p_x belonging to the point set

$$\prod_1^\infty V_n(X) - X,$$

for if this were not the case, then by virtue of (1) and equation (2.1) in which substitute X_n for X, Y_n for Y, 1 for i and 2 for r, it follows that $V_n(X) \supset M_2(X_n)\cdot M_1(Y_n) \supset X$; and if for each point p of $M_2(X_n)\cdot M_1(Y_n)$ we take a neighborhood N_p of p of diameter less than 1/4 the distance from p to the set of points $\overline{M_1(X_n)}+\overline{M_2 Y(_n)}$, and call $U_n(X)$ the sum of all the neighborhoods N_p, then it follows readily that

$$X \subset M_2(X_n)\cdot M_1(Y_n) \subset U_n(X) \subset \overline{U_n(X)} \subset V_n(X);$$

and hence $F[U_n(X)] \subset X_n+Y_n$, $U_n(X) \subset U_{n-1}(X)$ and $X=\prod_1^\infty \overline{U_n(X)}$; but then X has the potential order 2 in M relative to G, contrary to supposition.

Now if X and Y are any two elements of $H(a, b)$, $X\neq Y$, it follows that $p_x\neq p_y$. For suppose $X<Y$. Then since X contains a limit point of the sum of its successors in $E(a, b)$ but contains no limit point of $M_2(Y)$, it follows that there exist two elements Y_k and Y_m in the "Y-sequence" in (1) for the element X such that

$$X < Y_k < Y_m < Y;$$

and since Y contains a limit point of the sum of its predecessors in $E(a, b)$ but contains no limit point of $M_1(Y_m)$, it follows that there exists an element X_n of the "X-sequence" for Y in (1) such that

$$X < Y_k < Y_m < X_n < Y.$$

Consequently it follows with the aid of (2.2) that

$$p_x \subset M_1(Y_k) + Y_k \subset M_1(Y_m)$$

and

$$p_y \subset M_2(X_n) + X_n \subset M_2(Y_m),$$

and hence $p_x\neq p_y$.

Now let L denote the set of all points $[p_x]$ for all elements X of $H(a, b)$. Then L is uncountable and is an ordered set; indeed, it is only necessary to set $p_x<p_y$ when $X<Y$. Therefore by the Zarankiewicz lemma, there exists a point p_x of L which is a limit point both of its predecessors and of its followers, and hence both of $\sum X_n$ and of $\sum Y_n$, where the sequences $[X_n]$ and $[Y_n]$ satisfy (1). But $\sum X_n \subset M_1(X)$ and $\sum Y_n \subset M_2(X)$; and p_x must then belong either to $M_1(X)$ or to $M_2(X)$ and be a limit point of the other, contrary to the fact that these two sets are mutually separated. Thus the supposition that our theorem is false leads to a contradiction.

4. **Consequences of §3.** Let G be any uncountable non-separated collection of cuttings of M. Then since the product of any family $[\overline{U}_n]$ of closed sets is closed, §3 yields at once the result

(α) *All save a countable number of the elements of G are closed point sets.*

Now if X is any element of G such that $M-X$ is not the sum of two connected point sets, X cannot have a potential order 2 in M relative to G. For $M-X=M_1(X)+M_2(X)+M_3(X)$, where the sets $M_1(X)$, $M_2(X)$, and $M_3(X)$ are mutually separated and contain points a_1, a_2 and a_3 respectively; and if X had the potential order 2 relative to G, there would exist two elements X_1 and X_2 of G and a neighborhood U of X such that $F(U)\subset X_1+X_2$, $X_1\subset M_1(X)$, $X_2\subset M_2(X)$ and $\overline{U}\cdot(a_1+a_2+a_3)=0$; but then it would readily follow that the point set $M_3(X)\cdot(M-\overline{U})$ is non-vacuous and is both open and closed, contrary to the fact that M is connected. Thus in consequence of the theorem in §3 we have

(β) *The complement of each element of G, with the exception of a countable number of such elements, consists of exactly two components.*

Let us denote by ρ the property of any subset N of M not to be separated in M by any single element of G. Clearly each element X of G has the property ρ. We shall now show that

(γ) *All save a countable number of the elements of G are saturated in M relative to the property ρ.*

If, on the contrary, G contains an uncountable subcollection G_1 no element of which is saturated relative to the property ρ, then for each element Z of G_1 there exists at least one point p_z which is not separated from Z in M by any single element of G. Under these conditions it follows by the theorem and proof in §3 that there exist two points a and b of M and three elements Z, X and Y of $E(a, b)$ (the collection of all those elements of G_1 which separate a and b) such that $X<Z<Y$, and $M_2(X)\cdot M_1(Y)$ contains Z but does not contain the point p_z and also such that $X+Y$ does not contain p_z. But then by equation (2.1) we have either $p_z\subset M_1(X)$ or $p_z\subset M_2(Y)$. This is impossible because in the first case X separates p_z and Z in M and in the second case Y separates p_z and Z in M.

A cutting X of M is said to be an irreducible cutting of M provided that no proper subset of X is a cutting of M.

(δ) *All save a countable number of the elements of G are irreducible cuttings of M.*

If this is not so, there exists an uncountable collection G^0 of cuttings of M such that for each element X^0 of G^0 there exists an element X of G and a point p_x of X such that $X^0\subset X-p_x$. Since G is non-separated, it follows at once that G^0 is non-separated. Therefore by (γ) there exists an element X^0

of G^0 which is saturated relative to the property ρ defined by the collection G^0. Consequently there exists an element Y^0 of G^0 which separates X^0 and p_x in M, and one has $M-Y^0=M_1(Y^0)+M_2(Y^0)$, where $M_1(Y^0)\supset X^0$ and $M_2(Y^0)\supset p_x$. But then $M-Y=M_1(Y^0)\cdot(M-Y)+M_2(Y^0)\cdot(M-Y)$, and thus Y separates X in M (for $Y\cdot(X^0+p_x)\subset Y\cdot X=0$), which contradicts the non-separatedness of G.

We prove now the following general theorem:

THEOREM. *Every uncountable non-separated collection G of cuttings of a connected, metric, and separable space M contains a subcollection Q which contains all save possibly a countable number of the elements of G and such that each element X of Q has the following properties*: (a) *X is closed*; (b) *$M-X$ is the sum of two mutually separated connected point sets*; (c) *X is saturated in M relative to the property ρ defined by the collection Q, i.e., for every point p of $M-X$, there exists an element Y of Q which separates X and p in M*; (d) *X is an irreducible cutting of M*; *and* (e) *X has the potential order 2 in M relative to Q.*

To obtain the collection Q, let D be a countable subset of M which is dense in M and let us omit from G: (1) every element which does not possess each of the properties (a), (b), and (d); (2) every element which separates in M some pair of points a, b of D which are separated by only a countable number of elements of G; (3) every element which separates some pair a, b of points of D and contains no point p having the property that every neighborhood of p contains points of uncountably many distinct elements of G which separate a and b. Let G_1 denote the collection of the elements of G remaining after these omissions. Then by virtue of (α), (β) and (δ), together with the facts that there are only a countable number of pairs of points of D and that in the space M every uncountable set of points contains a point of condensation of itself, it follows that G_1 contains all save possibly a countable number of the elements of G.

Now let us omit from G_1 every element which is not saturated in M relative to the property ρ defined by the collection G_1 and also every element which does not have the potential order 2 in M relative to G_1. Let Q be the collection of elements of G_1 remaining after these omissions. Then Q contains all save a countable number of the elements of G_1 and hence also of G, and every element X of Q has the desired properties (a)-(e). Clearly X has properties (a), (b) and (d), for every element of G_1 has these properties. It remains to show that X has properties (c) and (e).

To show that X has property (c), let p be any point of $M-X$. There exists an element Y of G_1 which separates X and p, because every element of Q is saturated in M relative to the property ρ defined by G_1. Hence $M-Y$

$=M_1(Y)+M_2(Y)$, where $M_1(Y)\supset X$ and $M_2(Y)\supset p$. Also $M-X=M_1(X)+M_2(X)$, where $M_2(X)\supset Y$. Thus if a and b are points of $M_1(X)$ and $M_2(Y)$ respectively belonging to D, both X and Y separate a and b in M, and we have $X<Y$ in the order from a to b. Now there exists also an element Z of G_1 which separates X and Y in M, and it follows from §2 that Z also separates a and b in M, and we have the order $X<Z<Y$. Thus $Z\subset M_2(X)\cdot M_1(Y)$. Since X and Y are closed, $M_2(X)\cdot M_1(Y)$ is a neighborhood of Z, and hence it contains points of (and therefore contains all of) uncountably many elements of G which separate a and b in M. Therefore there exists at least one of these elements, say W, which belongs to Q, for all but a countable number of the elements of G belong to Q. Thus we have the order $X<W<Y$; and since $p\subset M_2(Y)$, it follows that W separates X and p in M. Consequently X has property (c).

Since X has the potential order 2 in M relative to G_1, there exist, as shown in §3, two points a and b of M such that X belongs to the collection $E(a, b)$ of all those elements of G_1 which separate a and b in M and such that there exist two sequences $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ of elements of $E(a, b)$ so that

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n$$

and such that if $U_n=M_2(X_n)\cdot M_1(Y_n)$, then $X=\prod_1^\infty \overline{U}_n$. Now for each n there exist, by virtue of property (c), two elements X_n' and Y_n' of Q belonging to $E(a, b)$ such that $X_n<X_n'<X<Y_n'<Y_n$. Hence if U_n' denotes the point set $M_2(X_n')\cdot M_1(Y_n')$, one has $U_n'\subset U_n$. Hence $X=\prod_1^\infty \overline{U}_n'$, and since $F(U_n')\subset X_n'+Y_n'$ and since clearly the sequence $[U_n]$ contains an infinite subsequence $[U_{n_i}]$ such that $U_{n_{i+1}}\subset U_{n_i}$, it follows that X has the potential order 2 in M relative to Q. This completes the proof.

5. **Decomposition of M by means of a non-separated collection G every element of which is saturated relative to property ρ.** Let G be any non-separated collection of subsets of M each of which is saturated in M relative to the property ρ defined by G. For each point e of M which belongs to no element of G, let E denote the point set consisting of e together with all points p of M which are not separated in M from e by any single element of G. Let S denote the collection whose elements are the elements of G together with all such point sets E thus defined. Clearly each element of S is closed and every point of M belongs to exactly one element of S. We shall show next that the collection S is non-separated.

Suppose, on the contrary, that some element X of S separates some pair of points p and q belonging to an element Y of S. Then $M-X=M_1(X)+M_2(X)$, where $M_1(X)\supset p$ and $M_2(X)\supset q$. Now by virtue of the definition of the collections G and S, it follows that there exists an element Z of G which separates X and p in M. Hence $M-Z=M_1(Z)+M_2(Z)$, where $M_1(Z)\supset X$ and $M_2(Z)\supset p$. Since Z belongs to G, it cannot separate Y in M; and therefore $p+q\subset Y\subset M_2(Z)$. But then

$$M-Z=[M_1(Z)+M_1(X)\cdot M_2(Z)]+M_2(X)\cdot M_2(Z),$$

and we have a separation of $M-Z$ into two mutually separated sets containing the points p and q respectively of Y, contrary to the fact that since Z belongs to G it cannot separate Y in M. Therefore S is non-separated.

Now clearly every element of S is saturated in M relative to the property ρ defined by the collection S. Consequently every two elements X and Y of S are separated in M by some third element of S. With the aid of this property it follows immediately that the collection S is upper semi-continuous,* i.e., there does not exist a sequence $X_1, X_2, X_3, \cdots$ of elements of S and two sequences $[p_i]$ and $[q_i]$ of points such that $p_i+q_i\subset X_i$ and which have sequential limit points p and q respectively belonging to two different elements P and Q respectively of S. For there exists an element X of S such that $M-X=M_1(X)+M_2(X)$ where $M_1(X)\supset P$ and $M_2(X)\supset Q$; and since for each i, X_i is a subset either of $M_1(X)$ or of $M_2(X)$, either $M_1(X)$ or $M_2(X)$ contains X_i for infinitely many i's; but this is impossible, for both p and q are limit points of every infinite subsequence of $X_1, X_2, X_3, \cdots$.

Now in case the space M is compact, the elements of S are closed and compact, and if for each pair of elements X and Y of S we define the distance $\rho(X, Y)$ between X and Y as the upper limit of the distances $\rho(x, y)$, where x and y are points of X and Y respectively, it readily follows that the space S' so obtained is compact, separable, metric and connected; and since it readily follows that every two "points" of S' are separated in S' by some third "point" of S', therefore† S' is an acyclic continuous curve.

6. Existence Theorem. *If the space M is connected im kleinen, there exists an uncountable non-separated collection of cuttings of M.*

Let a and b be any two points of M, and for each positive number r which is less than the distance from a to b, let $S(a, r)$ denote the set of all points of M whose distance from a is equal to r and let $I(a, r)$ denote the set of all points at a distance $<r$ from a. Then for each r, $S(a, r)$ separates a and b in M. Let $R(a, r)$ denote the component of $M-S(a, r)$ containing a, let $R(b, r)$ denote the component of $M-\overline{R(a, r)}$ containing b, and let X_r denote the point set $\overline{R(a, r)}\cdot\overline{R(b, r)}$. Then clearly X_r separates a and b in M and we have

(i) $$X_r\subset F[R(a, r)]\subset S(a, r), \text{ and } X_r=F[R(b, r)];$$

(ii) $$R(a, r)\subset I(a, r).$$

* See R. L. Moore, these Transactions, vol. 27 (1925), pp. 416–428.

† See R. L. Moore, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

Obviously the collection of cuttings $[X_r]$ is uncountable. It remains to show that it is non-separated. Let X_{r_1} and X_{r_2} be any two elements of this collection and suppose $r_1<r_2$. By (ii) it follows that $\overline{R(a, r_1)} \subset R(a, r_2)$. Thus $X_{r_1} \subset R(a, r_2)$, and therefore X_{r_2} does not separate X_r in M. From the inclusion $\overline{R(a, r_1)} \subset R(a, r_2)$ and (i) it follows that $\overline{R(b, r_2)} = R(b, r_2)+X_{r_2} \subset R(b, r_1)$, and consequently X_r does not separate X_{r_2} in M. Thus the collection $[X_r]$ is non-separated, and the theorem is proved.

Now since a may be any point whatever of M and since every neighborhood of a contains uncountably many of the sets $[X_r]$, it follows by §4, (δ), that every such neighborhood contains at least one set X_r which is an irreducible cutting of M. Thus we have the following

Corollary. *Every open subset of a connected and connected im kleinen point set M lying in a separable metric space contains an irreducible cutting I of M.*

This corollary answers a question raised by the author.*

As a result of this existence theorem it follows that the decomposition treated in §5 is always realisable in case M is locally connected; and in case M is a continuous curve, M may be decomposed upper semi-continuously into a collection S of the type attained in §5, and the decomposition space S' is a non-degenerate acyclic continuous curve.

7. **Concluding remarks.** Although it is easily seen with the aid of a very simple example that two cuttings X and Y of M may have the property that neither of them separates the other in M and yet the set $M_2(X)+X$ not be connected, where $M_2(X) \supset Y$, nevertheless the following lemma is true.

Lemma. *If a and b are two points of M and $X_1, X_2, X_3, \cdots$ is any infinite sequence of distinct mutually exclusive sets each of which separates a and b in M and no one of which separates any other one, and we have*

$$X_1 < X_2 < X_3 < \cdots,$$

then the set of points $\sum_1^\infty M_1(X_i)$ is connected.

For if on the contrary this set of points is the sum of two mutually separated sets N_1 and N_2, then since a belongs to all of the sets $M_1(X_i)$, there exists an integer n such that $N_1 \cdot M_1(X_n) \neq 0 \neq N_2 \cdot M_1(X_n)$. Since by (2.2), (b), it follows that $M_1(X_n) \subset M_1(X_{n+1})$, therefore $N_1 \cdot M_1(X_{n+1}) \neq 0 \neq N_2 \cdot M_1(X_{n+1})$. Since these two sets are mutually separated, one of them, say $N_1 \cdot M_1(X_{n+1})$,

* See Fundamenta Mathematicae, vol. 13 (1929), p. 50, where the question is raised for *continuous curves M*. A solution of this problem for the case where M is a plane continuous curve has been given by J. H. Roberts; see these Transactions, vol. 32 (1930), p. 19.

contains X_n. But then it is easily seen that the sets $N_2 \cdot M_1(X_n)$ and $M-N_2 \cdot M_1(X_n)$ are mutually separated, contrary to the fact that M is connected.

With the aid of this lemma it can be shown without great difficulty that if X is any element of a non-separated collection G of subsets of M which is saturated in M relative to the property ρ defined by the collection G, then

(1) each component of $M-X$ is open in M;

(2) the components of $M-X$ are countable;

(3) X is a potentially regular element of G in M relative to G, i.e., a monotone decreasing sequence of neighborhoods $[U_i]$ of X exists such that $F(U_i)$ is a subset of a finite number of the elements of G and $X=\prod_1^\infty \overline{U}_i$;

(4) the potential order of X in M relative to G is equal to the number of components of $M-X$ when this number is finite, and is equal to ω (i.e., X is of increasing order) when and only when this number is infinite.

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